Wiskunde 3 (Vector Analyse) 2002/2003

MidTaets®,

Zet op elk blad je naam en #

$$\text{Cijfer} = 1 + \frac{\text{aantal punten}}{2}$$

1) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = xy$

a) (2) Bepaal met de methode van Lagrange multiplicatoren de extremen van f beperkt tot $x + y^2 = 3$

b) (3) Bepaal de aard v.d. extremen met een tweede orde test.

2) a) (1) Schets het Domein

$$D = \{(x,y) \mid 4 \leq y + x^2 \leq 9,\ x \geq 0,\ y \geq 0\}.$$

b) (2) Bereken $\iint_D xy \, dx\, dy$.

3) (2) Gegeven een C^2 vector veld op $\mathbb{R}^3$.
Bewijs $\operatorname{div}(\operatorname{curl}(V)) = 0$

4) Laat $S = \{(x,y,z) \mid x^2 + y^2 = 1 + z^2\}$

a) (1) Schets $S \cap \{(x,y,z) \mid y = 0\}$

b) (2) Bereken het volume van

$$\begin{cases} x^2 + y^2 \leq 1 + z^2 \\ 0 \leq z \leq 2 \end{cases}$$

c) (3) Laat $V = \{(x,y,z) \mid \sqrt{2}\, z = x + y\}$.
De Doorsnede van S en V tussen $z = 0$ en $z = 2$ bestaat uit twee krommen.
Beschrijf beide krommen als een pad $c_1(t)$ en $c_2(t)$

d) (2) Bepaal de lengte van beide paden.

1a. $f: \mathbb{R}^2 \to \mathbb{R}$

2 $f(x, y) = xy$

$g(x, y) = x + y^2 - 3$

$g = 0$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$

$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (1, 2y)$

$\nabla f = \lambda \nabla g$

$(y, x) = (\lambda, 2\lambda y)$

$$\left.\begin{array}{l} y = \lambda \\ x = 2\lambda y \\ x + y^2 - 3 = 0 \end{array}\right\}$$

$\cancel{x = 2\lambda y = 2\lambda^2}$

$x + y^2 - 3 = x + \lambda^2 - 3 = 2\lambda^2 + \lambda^2 - 3 =$ $\qquad 3\lambda^2 - 3 = 0 \to \lambda^2 = 1$

$\lambda = -1$ | $\lambda = 1$
$y = -1$ | $y = 1$
$x = 2$ | $x = 2$

$(2, -1)$ is een kritiek punt

$(2, 1)$ is een kritiek punt

b.

2 $f - \lambda g = xy - \lambda x - \lambda y^2 + 3\lambda$

$$H_{f-\lambda g} = \begin{pmatrix} \frac{\partial^2 (f-\lambda g)}{\partial x^2} & \frac{\partial^2 (f-\lambda g)}{\partial x \partial y} \\ \frac{\partial^2 (f-\lambda g)}{\partial y \partial x} & \frac{\partial^2 (f-\lambda g)}{\partial y^2} \end{pmatrix}$$

$\frac{\partial (f-\lambda g)}{\partial x} = y - \lambda$

$\frac{\partial (f-\lambda g)}{\partial y} = x - 2\lambda y$

$\frac{\partial^2 (f-\lambda g)}{\partial x \partial y} = \frac{\partial^2 (f-\lambda g)}{\partial y \partial x} = 1$

$\frac{\partial^2 (f-\lambda g)}{\partial x^2} = 0$

$\frac{\partial^2 (f-\lambda g)}{\partial y^2} = -2\lambda$

$$H_{f-\lambda g} = \begin{pmatrix} 0 & 1 \\ 1 & -2\lambda \end{pmatrix}$$

$\lambda = -1$

$$H_{f-\lambda g} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$\lambda = 1$

$$H_{f-\lambda g} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

Je moet kijken naar de beperkte of geranderde Hessian

Beide Hessianen zijn niet positief of negatief definiet, want het element linksboven is 0. De test geeft hier geen uitsluitsel.

Wat wel kan is de restrictie g als kromme te schrijven:

$\varphi(t) = (3 - t^2, t)$

$f(\varphi(t)) = (3 - t^2)(t) = 3t - t^3$

$f'(t) = 3 - 3t^2$ (voor $t = -1$ en $t = 1$, dus $(2,-1)$ resp $(2,1)$ is deze inderdaad 0)

$f''(t) = -6t$

$f''(-1) = 6 > 0$ dus $(2,-1)$ is een minimum, $f''(1) = -6 < 0$ dus $(2,1)$ is een maximum } van xy beperkt tot $x + y^2 = 3$

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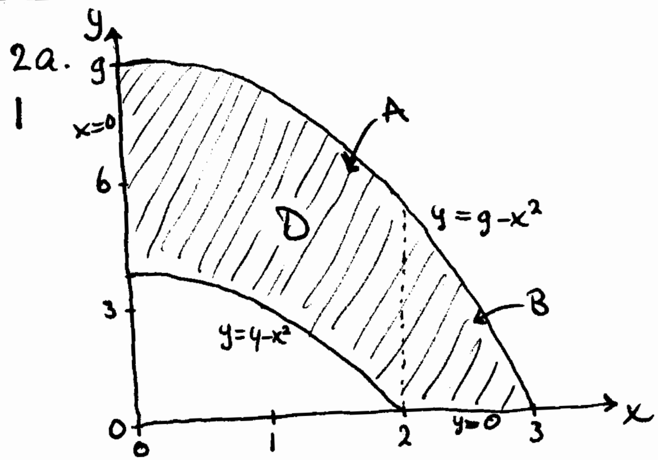
2a.

1



$4 \leq y + x^2$

$4 - y \leq x^2$

$-y \leq x^2 - 4$

$y \geq 4 - x^2$

$A = D \cap \{(x,y) \mid x \leq 2\}$

$B = D \cap \{(x,y) \mid x > 2\}$

$y + x^2 \leq 9$

$y \leq 9 - x^2$

b. $\iint_D xy\,dx\,dy = \iint_A xy\,dx\,dy + \iint_B xy\,dx\,dy$

2

$\iint_A xy\,dx\,dy = \int_0^2 \left(\int_{4-x^2}^{9-x^2} xy\,dy\right) dx = \int_0^2 \left[\frac{1}{2}xy^2\right]_{y=4-x^2}^{y=9-x^2} dx = \int_0^2 \left(\frac{1}{2}x(9-x^2)^2 - \frac{1}{2}x(4-x^2)^2\right) dx =$

$\int_0^2 \left(\frac{1}{2}x\left(9^2 - 18x^2 + x^4 - 4^2 + 8x^2 - x^4\right)\right) dx = \int_0^2 \frac{1}{2}x(-10x^2 + 65)\,dx = \int_0^2 \left(-5x^3 + 32\tfrac{1}{2}x\right) dx =$

$\left[-\frac{5}{4}x^4 + 16\tfrac{1}{4}x^2\right]_{x=0}^{x=2} = -\frac{5}{4}\cdot 2^4 + 16\tfrac{1}{4}\cdot 2^2 = -\frac{5}{4}\cdot 16 + 16\tfrac{1}{4}\cdot 4 = -20 + 65 = 45$

$\iint_B xy\,dx\,dy = \int_2^3 \left(\int_0^{9-x^2} xy\,dy\right) dx = \int_2^3 \left[\frac{1}{2}xy^2\right]_{y=0}^{y=9-x^2} dx = \int_2^3 \frac{1}{2}x(9-x^2)^2\,dx =$

$\int_2^3 \frac{1}{2}x(81 - 18x^2 + x^4)\,dx = \int_2^3 \left(40\tfrac{1}{2}x - 9x^3 + \frac{1}{2}x^5\right) dx = \left[20\tfrac{1}{4}x^2 - \frac{9}{4}x^4 + \frac{1}{12}x^6\right]_{x=2}^{x=3} =$

$60\tfrac{3}{4} - 50\tfrac{1}{3} = 10\tfrac{5}{12}$

$\iint_D xy\,dx\,dy = \iint_A xy\,dx\,dy + \iint_B xy\,dx\,dy = 45 + 10\tfrac{5}{12} = 55\tfrac{5}{12}$

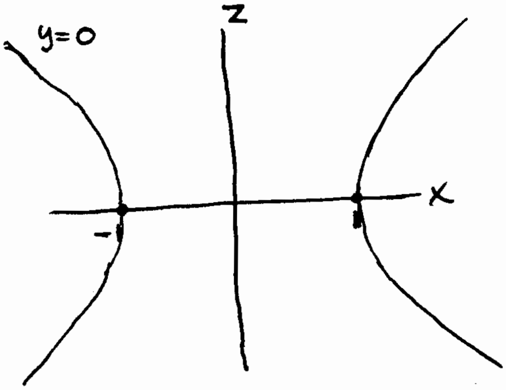
3. $V: \mathbb{R}^3 \to \mathbb{R}^3$

$\text{curl}(V) = V \times \nabla$ $\qquad$ ∇ is geen vector $\in \mathbb{R}^3$

$\text{div}(\text{curl}(V)) = (V \times \nabla) \cdot \nabla$; het uitproduct $\times$ staat loodrecht op beide vectoren:

0 $(V \times \nabla) \perp (\nabla) \Rightarrow (V \times \nabla) \cdot \nabla = 0$ want het inproduct $\cdot$ van loodrechte vectoren is 0. Q.E.D.

4a. $S \cap \{(x,y,z) \mid y=0\} = \{(x,y,z) \mid x^2+y^2 = 1+z^2 \wedge y=0\} = \{(x,y,z) \mid x^2 = 1+z^2\} \wedge y=0\}$



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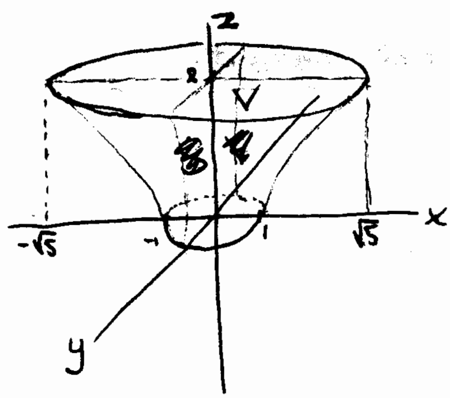
b.



$1+z^2 > 0 \ \forall z$

Dit is een omwentelingslichaam van $x \leq \sqrt{1+z^2}$, $z \in [0,2]$

2 $\text{Vol}(V) = \pi \int_0^2 x^2 \, dz = \pi \int_0^2 \sqrt{1+z^2}^2 \, dz = \pi \int_0^2 (1+z^2) \, dz = \pi \left[z + \frac{1}{3}z^3\right]_0^2 = \pi\left(2 + \frac{8}{3}\right) = \frac{14}{3}\pi$

c. $V = \{(x,y,z) \mid z = \frac{x+y}{\sqrt{2}}\}$

$S = \{(x,y,z) \mid x^2+y^2 = 1+z^2\}$

$\forall (x,y,z) \in (V \cap S)$:

$z = \frac{x+y}{\sqrt{2}}$

$x^2+y^2 = 1+z^2$

$x^2+y^2 = 1 + \left(\frac{x+y}{\sqrt{2}}\right)^2 = 1 + \frac{(x+y)^2}{2} = 1 + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$

$\frac{1}{2}x^2 + \frac{1}{2}y^2 = 1 + xy$

$x^2+y^2 = 2 + 2xy$

$x^2+y^2-2xy = 2$

$(x-y)^2 = 2$

2.0.2.

c. (vervolg)

$c_k(t) = (c_{k,1}(t), c_{k,2}(t), c_{k,3}(t))$

$c_{k,3} = \dfrac{c_{k,1} + c_{k,2}}{\sqrt{2}}$

$(c_{k,1} - c_{k,2})^2 = 2$

$c_{1,1} - c_{1,2} = \sqrt{2}$

$c_{1,2} = c_{1,1} - \sqrt{2}$

Neem $c_{1,1}(t) = t$:

$c_{1,2}(t) = t - \sqrt{2}$

$c_{1,3}(t) = \dfrac{2t - \sqrt{2}}{\sqrt{2}} = \sqrt{2}\,t - 1$

$c_1(t) = (t, t - \sqrt{2}, \sqrt{2}\,t - 1)$

$c_{2,1} - c_{2,2} = -\sqrt{2}$

$c_{2,2} = c_{2,1} + \sqrt{2}$

Neem $c_{2,1}(t) = t$:

$c_{2,2}(t) = t + \sqrt{2}$

$c_{2,3}(t) = \dfrac{2t + \sqrt{2}}{\sqrt{2}} = \sqrt{2}\,t + 1$

$c_2(t) = (t, t + \sqrt{2}, \sqrt{2}\,t + 1)$

3

d. Lengte $(c_1) = \displaystyle\int_{t_0}^{t_1} \sqrt{(c_{1,1}'(t))^2 + (c_{1,2}'(t))^2 + (c_{1,3}'(t))^2}\,dt$

met $c_1(t_0)$ het beginpunt ($z=0$) en $c_1(t_1)$ het eindpunt. ($z=2$)

$c_{1,3}(t_0) = 0 \to \sqrt{2}\,t_0 - 1 = 0 \to \sqrt{2}\,t_0 = 1 \to t_0 = \frac{1}{2}\sqrt{2}$

$c_{1,3}(t_1) = 2 \to \sqrt{2}\,t_1 - 1 = 2 \to \sqrt{2}\,t_1 = 3 \to t_1 = \frac{3}{\sqrt{2}} = 3 \cdot \frac{1}{\sqrt{2}} = 3 \cdot \frac{1}{2}\sqrt{2} = \frac{3}{2}\sqrt{2}$

Lengte$(c_1) = \displaystyle\int_{\frac{1}{2}\sqrt{2}}^{\frac{3}{2}\sqrt{2}} \sqrt{1^2 + 1^2 + \sqrt{2}^2}\,dt = \int_{\frac{1}{2}\sqrt{2}}^{\frac{3}{2}\sqrt{2}} \sqrt{4} = \Big[2t\Big]_{\frac{1}{2}\sqrt{2}}^{\frac{3}{2}\sqrt{2}} = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$

$c_{2,3}(t_0) = 0 \to \sqrt{2}\,t + 1 = 0 \to \sqrt{2}\,t = -1 \to t_0 = -\frac{1}{\sqrt{2}} = -\frac{1}{2}\sqrt{2}$

$c_{2,3}(t_1) = 2 \to \sqrt{2}\,t_1 + 1 = 2 \to \sqrt{2}\,t_1 = 1 \to t_1 = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$

Lengte$(c_2) = \displaystyle\int_{-\frac{1}{2}\sqrt{2}}^{\frac{1}{2}\sqrt{2}} \sqrt{1^2 + 1^2 + \sqrt{2}^2}\,dt = \int_{-\frac{1}{2}\sqrt{2}}^{\frac{1}{2}\sqrt{2}} \sqrt{4}\,dt = \Big[2t\Big]_{-\frac{1}{2}\sqrt{2}}^{\frac{1}{2}\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

2